

**Elementary treatment of  $p^a \pm p^b + 1 = x^2$** 

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**Abstract:** We give a shorter simpler proof of a result of Szalay on the equation  $2^a + 2^b + 1 = x^2$ . We give an elementary proof of a result of Luca on the equation of the title for prime  $p > 2$ . The elementary treatment is made possible by a lemma which is also used to obtain a bound on  $n$  in the equation  $x^2 + C = y^n$ , where  $x$  and  $y$  are primes or prime powers, and  $2|C$ ; the bound depends only on the primes dividing  $C$ .

## §1 Introduction

All solutions to the title equation for prime  $p$  and positive integers  $a$ ,  $b$ , and  $x$  have been found by Szalay [Sz] and Luca [Lu]. For the case  $p = 2$ , Szalay uses a result of Beukers [Bk]. For the case  $p > 2$ , Luca's proof includes sections which use lower bounds on linear forms in logarithms (see [Lu, pp. 7–11]) and the well known recent work of Bilu, Hanrot, and Voutier [BHV] (see [Lu, pp. 12–14]).

The purpose of this paper is to give shorter and simpler proofs of these results. In particular, for the case  $p > 2$ , we can make the treatment completely elementary, eliminating the sections using linear forms in logarithms and the results of [BHV].

In Section 2, we obtain a shorter proof of Szalay's result for the case  $2^a + 2^b + 1 = x^2$  by replacing the older bound in [Bk] with the recent sharp result of Bauer and Bennett [BB], not available to Szalay. Szalay's proof can be further shortened by observing that the methods of his Lemma 8 alone suffice to give the desired contradiction to Bauer and Bennett's results; the remaining auxiliary results in [Sz], including the mapping of one set of solutions onto another, are not needed. Szalay has already obtained a short proof in the case  $2^a - 2^b + 1 = x^2$ , since here a result of Beukers [Bk] on the number of solutions to the equation  $x^2 + D = 2^n$  applies; so we will not treat this case here. Also, we will not treat the cases  $2^a \pm 2^b - 1 = x^2$  which require  $b = 1$  and are thus easily solved (see [Sz, Theorem 3]).

In Section 3 we obtain a short and elementary proof of Luca's results, without interfering with the clever use of continued fractions in [Lu], by

using two elementary lemmas which replace the use of linear forms in logarithms and [BHV]. The second of these, Lemma 3.2, has a further application given in Section 4: we establish a bound on  $n$  in the equation  $x^2 + C = y^n$  when  $x$  and  $y$  are primes or prime powers and  $2|C$ . The bound depends only on the primes dividing  $C$ . Beukers [Bk] established a bound on  $n$  for more general  $x$  when  $y = 2$ , and Bauer and Bennett [BB] greatly improved this bound as well as allowing  $y$  to take on many specific values. The bounds of [Bk] and [BB] depend on the value of  $y$  and the specific value of  $C$ . See also earlier results of Nagell [N] and Ljunggren [Lj].

While our new versions of the proofs of Theorems 1.5 and 1.6 below render elementary what was previously not elementary, Luca's proof of Theorem 1.4 depends only on work of Carmichael [Car]; we have removed the dependence on [Car]. Gary Walsh pointed out to the author that [Car] is not needed for proving the lemma [Lu, pg. 5] used by Luca to obtain Theorem 1.5; this lemma, however, is not used in our proof of Theorem 1.5.

In spite of these various changes, the basic approaches in the proofs given here parallel [Sz] and [Lu]: as in [Sz], we show that  $a$  is large enough relative to  $b$  to give a contradiction to a result derived from the theory of hypergeometric functions; as in [Lu], we use ideal factorization in  $\mathbf{Q}(\sqrt{1 \pm p^b})$ , or, in the case of Theorem 1.4 below, the basic theory of Pell equations.

The relevant results of Szalay and Luca are the following:

**Theorem 1.1 (Szalay)**      The equation

$$2^a + 2^b + 1 = x^2 \tag{1.1}$$

has no solutions in positive integers  $(a, b, x)$  with  $a \geq b$  except for the following cases:

$$(a, b, x) = (2t, t + 1, 2^t + 1) \text{ for positive integer } t \tag{A}$$

$$(a, b, x) = (5, 4, 7) \tag{B}$$

$$(a, b, x) = (9, 4, 23) \tag{C}$$

**Theorem 1.2 (Szalay)**      The equation

$$2^a - 2^b + 1 = x^2 \tag{1.2}$$

has no solutions in positive integers  $(a, b, x)$  with  $a > b$  except for the following cases:

$$(a, b, x) = (2t, t + 1, 2^t - 1) \text{ for positive integer } t > 1 \tag{D}$$

$$(a, b, x) = (5, 3, 5) \tag{E}$$

$$(a, b, x) = (7, 3, 11) \tag{F}$$

$$(a, b, x) = (15, 3, 181) \tag{G}$$

**Theorem 1.3 (Luca)** The only solutions of the equation

$$p^a \pm p^b + 1 = x^2 \tag{1.3}$$

in positive integers  $(x, p, a, b)$ , with  $a > b$ , and  $p$  an odd prime number are  $(x, p, a, b) = (5, 3, 3, 1), (11, 5, 3, 1)$ .

Luca divides this theorem into three subsidiary theorems:

**Theorem 1.4 (Luca)** The equation

$$x^2 = y^a + \varepsilon_1 y^b + \varepsilon_2, \quad \varepsilon_1, \varepsilon_2 \in \{1, -1\}, \tag{1.4}$$

has no positive integer solutions  $(x, y, a, b)$  with  $a > b$ ,  $a$  even, and  $y > 2$  and not a perfect power of some other integer.

**Theorem 1.5 (Luca)** There are no solutions to the equation

$$p^a + p^b + 1 = x^2 \tag{1.5}$$

in positive integers  $(x, p, a, b)$  with  $a$  odd and  $p$  an odd prime.

**Theorem 1.6 (Luca)** The only solutions to the equation

$$p^a - p^b + 1 = x^2 \tag{1.6}$$

in positive integers  $(x, p, a, b)$  with  $a > b$  and  $p$  an odd prime are  $(x, p, a, b) = (5, 3, 3, 1), (11, 5, 3, 1)$ . ■

We would like to thank Michael Bennett for calling our attention to Szalay's paper, Gary Walsh for calling our attention to Luca's paper, and Robert Styer for invaluable suggestions and assistance in preparing this paper.

## §2 A shortened proof of Szalay's result

Proof of Theorem 1.1:

Assume (1.1) has a solution that is not one of (A), (B), or (C). It is an easy elementary result that the only solution to (1.1) with  $a = b$  is given by Case (A) with  $t = 1$ , so we can assume hereafter  $a > b$ .

Considering (1.1) modulo 8, we get  $b > 2$ . If  $b = 3$ , then  $2^a = x^2 - 2^3 - 1 = (x + 3)(x - 3)$ , giving  $x = 5$ , which is Case (A) with  $t = 2$ , so we can assume hereafter  $b > 3$ .

Write  $x = 2^t k \pm 1$  for  $k$  odd and the sign chosen to maximize  $t > 1$ . In what follows, we will always take the upper sign when  $x \equiv 1 \pmod{4}$  and the lower sign when  $x \equiv 3 \pmod{4}$ .

We have

$$2^a + 2^b + 1 = 2^{2t} k^2 \pm (k \mp 1) 2^{t+1} + 2^{t+1} + 1. \quad (2.1)$$

From this we see  $b = t + 1$  so that  $t \geq 3$ . Now (2.1) yields  $a \geq 2t - 1$  with equality only when  $t = 3$ ,  $k = 1$ , and  $x \equiv 3 \pmod{4}$ , which is Case (B), already excluded. So  $a \geq 2t$ , hence  $a > 2t$  since Case (A) has been excluded. So now

$$k \mp 1 = 2^{t-1} g \text{ for some odd } g > 0$$

We have

$$2^{a-2t} = k^2 \pm g = 2^{2t-2} g^2 \pm 2^t g + 1 \pm g \quad (2.2)$$

(2.2) yields  $a - 2t \geq 2t - 3$  with equality only when  $t = 3$ ,  $g = 1$ , and  $x \equiv 3 \pmod{4}$ , which is Case (C), already excluded. So now  $g \pm 1 = 2^t h$  for some odd  $h > 0$ . So we must have  $g \geq 2^t \mp 1$ . Assume  $x \equiv 3 \pmod{4}$ . Then from (2.2) we derive

$$2^{a-2t} > g^2(2^{2t-2} - 1) > 2^{2t} 2^{2t-3} = 2^{4t-3} \quad (2.3)$$

Now assume  $x \equiv 1 \pmod{4}$ . Then

$$2^{a-2t} > 2^{2t-2} g^2 \geq 2^{2t-2} (2^{2t} - 2^{t+1} + 1) > 2^{2t-2} 2^{2t-1} = 2^{4t-3}$$

In both cases we have

$$a \geq 6t - 2 = 6b - 8 \quad (2.4)$$

Now we can use Corollary 1.7 in Bauer and Bennett [BB]:

$$a < \frac{2}{2 - 1.48} \frac{\log(2^b + 1)}{\log(2)}$$

Thus,

$$a < \frac{1}{0.26} \frac{\log(2^b + 1)}{\log(2^b)} b < \frac{1}{0.26} \frac{\log(17)}{\log(16)} b < 4b$$

Combining this with (2.4) we obtain  $b < 4$  which is impossible since  $b > 3$ . This completes the proof of Theorem 1.1.

A similar treatment handles Theorem 1.2, although here we must use the familiar results on the equation  $x^2 + 7 = 2^y$  to handle the case  $b = 3$ ,

and also use a slightly more refined computation to establish the second inequality of (2.3), which here applies to  $x \equiv 1 \pmod{4}$ . As pointed out in the introduction, however, Szalay already has a short proof of Theorem 1.2.

### §3 Elementary proofs of Luca's results

Proof of Theorem 1.4: First we consider the case  $b$  even. We establish some notation as in [Lu]. Letting  $X = x$ ,  $Y = y^{b/2}$ , and  $D = y^{a-b} + \varepsilon_1$ , we rewrite (1.4) as

$$X^2 - DY^2 = \varepsilon_2. \quad (3.1)$$

The least solution of  $U^2 - DV^2 = \pm 1$  is  $(U, V) = (y^{(a-b)/2}, 1)$ . Write  $X_n + Y_n\sqrt{D} = (y^{(a-b)/2} + \sqrt{D})^n$  for any integer  $n$ . For some  $j > 1$ ,  $(X, Y) = (X_j, Y_j)$ . As in [Lu], it is easily seen that  $2|j$ . At this point we diverge from [Lu] and apply Lemmas 1–3 of [Sc] to see that, if  $j > 2$ , there exists a prime  $q$  such that  $q|y$ ,  $q|(Y_j/Y_2)$ ,  $Y_{2q}|Y_j$ , and  $Y_{2q}/(qY_2)$  is an integer prime to  $y$ . But since  $Y_{2q}/(qY_2)$  is greater than 1 and divides  $Y_j$ , we have a contradiction. So  $j = 2$  and we must have

$$y^{b/2} = Y = Y_2 = 2y^{(a-b)/2}. \quad (3.2)$$

Now we consider the case  $b$  odd and again establish notation as in [Lu]. Letting  $X = x$ ,  $Y = y^{(b-1)/2}$ , and  $D = y(y^{a-b} + \varepsilon_1)$ , we rewrite (1.4) as (3.1). At this point we diverge from [Lu] and apply an old theorem of Störmer [Stö]: his Theorem 1 says if every prime divisor of  $Y$  divides  $D$  in (3.1), then  $(X, Y) = (X_1, Y_1)$ , the least solution of (3.1). Theorem 1 of [Stö] also applies to show that  $(2y^{a-b} + \varepsilon_1, 2y^{(a-b-1)/2})$  is the least solution  $(U_1, V_1)$  of  $U^2 - DV^2 = 1$ . If  $\varepsilon_2 = -1$ , then  $2X_1Y_1 = 2y^{(a-b-1)/2}$ , which is impossible since  $(X_1, y) = 1$ , and  $y > 2$  implies  $x = X_1 > 1$ . Thus we must have  $\varepsilon_2 = 1$ , so that

$$y^{(b-1)/2} = Y = Y_1 = V_1 = 2y^{(a-b-1)/2}. \quad (3.3)$$

At this point we return to [Lu] where it is pointed out that (3.2) and (3.3) require  $y = 2$  which is not under consideration. This completes the proof of Theorem 1.4.

We note that Theorem 1 of [Stö] has a short elementary proof.

For Theorems 1.5 and 1.6 we will need the following:

**Lemma 3.1** Let  $D$  be any squarefree integer, let  $u$  be a positive integer, and let  $S$  be the set of all numbers of the form  $r + s\sqrt{D}$ , where  $r$  and  $s$  are nonzero rational integers,  $(r, sD) = 1$ , and  $u|s$ . Let  $p$  be any

odd prime number, and let  $t$  be the least positive integer such that  $\pm p^t$  is expressible as the norm of a number in  $S$ , if such  $t$  exists. Then, if  $\pm p^n$  is also so expressible, we must have  $t|n$ . (Note the  $\pm$  signs in the statement of this lemma are independent.)

Proof: Assume that for some  $p$  and  $S$ , there exists  $t$  as defined in the statement of the lemma. Then  $p$  splits in  $\mathbf{Q}(\sqrt{D})$ ; let  $[p] = PP'$ . For each positive integer  $k$  there exists an  $\alpha$  in  $S$  such that  $P^{kt} = [\alpha]$ . Now suppose  $\pm p^{kt+g}$  equals the norm of  $\gamma$  in  $S$  where  $k$  and  $g$  are positive integers with  $g < t$ . Since  $P^{kt+g}$  must be principal,  $P^g = [\beta]$  for some irrational integer  $\beta \in \mathbf{Q}(\sqrt{D})$ . Therefore, for some unit  $\epsilon$ , either  $\gamma = \epsilon\alpha\beta$  or  $\bar{\gamma} = \epsilon\alpha\beta$ .  $\epsilon\alpha\beta$  has integer coefficients and the norm of  $\alpha$  is odd, so  $\epsilon\beta$  has integer coefficients. Now  $\alpha \in S$  and  $\epsilon\alpha\beta \in S$ , so that one can see that  $\epsilon\beta \in S$ , which is impossible by the definitions of  $t$  and  $g$ .

Proof of Theorem 1.5: We first establish some notation by paraphrasing [Lu, Section 3]: Looking at (1.5), we see that the only case in which solutions with odd  $a$  might exist is when  $p \equiv 3 \pmod{4}$  and  $b$  is even; let  $p^b + 1 = Du^2$ , with  $D$  square-free and  $u > 0$  an integer. At this point we diverge from [Lu] and note that if  $S$  is the set of all integers of the form  $r + s\sqrt{D}$  with nonzero rational integers  $r$  and  $s$ ,  $(r, sD) = 1$  and  $u|s$ , then  $p^a$  and  $-p^b$  are both expressible as the norms of numbers in  $S$ . Therefore Lemma 3.1 shows that  $\pm p^c$  is expressible as the norm of a number in  $S$ , where  $c$  divides both  $a$  and  $b$ . From this point on, we return to the method of proof of [Lu]:  $a$  is odd and  $b$  is even, so we have  $c \leq b/2$ . For some coprime positive integers  $v$  and  $w$  such that  $(v, p^b + 1) = 1$ , we must have

$$v^2 - w^2(p^b + 1) = \pm p^c. \quad (3.4)$$

(3.4) corresponds to (17) in [Lu]. Since  $|p^c| < \sqrt{p^b + 1}$ ,  $v/w$  must be a convergent of the continued fraction for  $\sqrt{p^b + 1}$ . But then, since  $p^b + 1$  is of the form  $m^2 + 1$ , we must have  $p^c = \pm 1$ , impossible, finishing the proof of Theorem 1.5.

Proof of Theorem 1.6: As in [Lu], we write  $p^b - 1 = Du^2$ ,  $D$  and  $u$  positive integers and  $D$  squarefree, and consider the equation

$$p^n = h^2 + k^2 u^2 D \quad (3.5)$$

in relatively prime nonzero integers  $h$  and  $k$ , and positive integer  $n$ . From (1.6) we see that (3.5) has the solutions  $(n, h, k) = (b, 1, 1)$  and  $(a, x, 1)$ . Clearly,  $p$  splits in  $\mathbf{Q}(\sqrt{-D})$ , and we can let  $[p] = \pi_1 \pi_2$  be its factorization into ideals. We can take

$$\pi_1^b = [1 + u\sqrt{-D}], \pi_1^a = [x \pm u\sqrt{-D}]. \quad (3.6)$$

At this point we diverge from [Lu]: clearly  $b$  is the least possible value of  $n$  in (3.5), so we can apply Lemma 3.1 to obtain  $b|a$ . Thus,

$$(1 + u\sqrt{-D})^{a/b} = (x \pm u\sqrt{-D})\epsilon \quad (3.7)$$

where  $\epsilon$  is a unit in  $\mathbf{Q}(\sqrt{-D})$ . If  $D = 1$  or  $3$ , we note  $2|u$  and  $2 \nmid x$ , so that we must have  $\epsilon = \pm 1$ . Thus, using (3.7), we see that Theorem 1.6 follows immediately upon establishing the following elementary lemma:

**Lemma 3.2**      The equation

$$(1 + \sqrt{-D})^r = a \pm \sqrt{-D} \quad (3.8)$$

has no solutions with  $r > 1$  when  $D$  is a positive integer congruent to  $2 \pmod{4}$  and  $a$  is any integer, except for  $D = 2$ ,  $r = 3$ .

Further, when  $D$  congruent to  $0 \pmod{4}$  is a positive integer such that  $1 + D$  is prime or a prime power, (3.8) has no solutions with  $r > 1$  except for  $D = 4$ ,  $r = 3$ .

(Note that here  $D$  corresponds to  $Du^2$  in the proof of Theorem 1.6 above, which follows the notation of [Lu]. Thus, in the proof of Lemma 3.2,  $D$  is not necessarily squarefree. Note also  $r$  corresponds to  $a/b$  in the proof of Theorem 1.6.)

Proof of Lemma 3.2: Assume (3.8) has a solution with  $r > 1$  for some  $a$  and  $D$ . From Theorem 13 of [BH], we see that, if  $r > 1$ , then  $r$  is a prime congruent to  $3 \pmod{4}$  and there is at most one such  $r$  for a given  $D$ . Thus we obtain

$$(-1)^{\frac{D+2}{2}} = r - \binom{r}{3}D + \binom{r}{5}D^2 - \dots - D^{\frac{r-1}{2}} \quad (3.9)$$

If  $r = 3$ , (3.9) shows that  $|D - 3| = 1$ , giving the two exceptional cases of the Lemma. So from here on we assume  $3 \nmid r$ .

We will use two congruences:

Congruence 1 : 
$$(-1)^{\frac{D+2}{2}} \equiv \left(\frac{r}{3}\right) 2^{r-1} \pmod{D-3}$$

Congruence 2 : 
$$(-1)^{\frac{D+2}{2}} \equiv 2^{r-1} \pmod{D+1}$$

Congruences 1 and 2 correspond to congruences (9e) and (9f) of Lemma 7 of [BH]. From Congruence 1 we see that  $D - 3$  cannot be divisible both by a prime  $3 \pmod{4}$  and a prime  $5 \pmod{8}$ . So  $D \equiv 2 \pmod{4}$  implies  $D \not\equiv 3 \pmod{5}$ . Now let  $D + 1 = y$ . If  $D \equiv 1 \pmod{5}$ ,  $y^r \equiv 3 \pmod{5}$ ; since

$a^2 + D = y^r$ ,  $a^2 \equiv 2 \pmod{5}$ , impossible. If  $D \equiv 2 \pmod{5}$ ,  $y^r \equiv 2 \pmod{5}$ , so that 5 divides  $a$ . Since in this case  $D$  is a quadratic nonresidue modulo 5, we see from (3.8) that  $5|a$  implies  $3|r$ , which we have excluded. Now  $y^r$  is congruent to  $-y$  modulo  $y^2 + 1$  so that  $a^2$  is congruent  $-2y + 1$  modulo  $y^2 + 1$ . So, using the Jacobi symbol, we must have

$$1 = \left( \frac{-2y + 1}{(y^2 + 1)/2} \right) = \left( \frac{2y^2 + 2}{2y - 1} \right) = \left( \frac{y + 2}{2y - 1} \right) = \left( \frac{-5}{y + 2} \right)$$

If  $D \equiv 2 \pmod{4}$ , then  $y \equiv 3 \pmod{4}$  and the last Jacobi symbol in this sequence equals  $\left( \frac{y+2}{5} \right) = \left( \frac{D+3}{5} \right)$ , which has the value -1 when  $D$  is congruent to 0 or 4 modulo 5. Thus, when  $D \equiv 2 \pmod{4}$ , we have shown that there are no values of  $D$  modulo 5 that are possible.

So we assume hereafter that  $D \equiv 0 \pmod{4}$ . Write  $D + 1 = p^n$  where  $p$  is prime, and let  $g$  be the least number such that  $2^g \equiv -1 \pmod{p}$ , noting Congruence 2. We see that  $g|r - 1$  and also  $g|p - 1|p^n - 1 = D$ . Now (3.9) gives  $-1 \equiv 1 \pmod{g}$  so that  $g \leq 2$ . Assume first that  $n$  is odd. Since  $4|D$ ,  $p \equiv 1 \pmod{4}$ . In this case, we must have  $g = 2$ ,  $p = 5$ . If  $n$  is even, since we have  $1 + D = p^n$  and  $a^2 + D = p^{rn}$ , we must have  $2p^{rn/2} - 1 \leq D = p^n - 1$ , giving  $r < 2$ , impossible. So we have  $n$  odd,  $p = 5$ .

Since  $n$  is odd,  $D \equiv 4 \pmod{8}$ , and, since  $\binom{r}{3}$  is odd, (3.9) gives  $r \equiv 3 \pmod{8}$ . Now assume  $r \equiv 2 \pmod{3}$  and let  $y = 5^n = 1 + D$ . Then  $y^r \equiv y^2 \pmod{y^3 - 1}$ , so that  $a^2 \equiv y^2 - y + 1 \pmod{y^2 + y + 1}$ , so that

$$1 = \left( \frac{y^2 - y + 1}{y^2 + y + 1} \right) = \left( \frac{-2y}{y^2 + y + 1} \right) = \left( \frac{-2}{y^2 + y + 1} \right)$$

which is false since  $y^2 + y + 1 \equiv 7 \pmod{8}$ . Thus we have  $r \equiv 19 \pmod{24}$  so that  $y^r \equiv -y^7 \pmod{y^{12} + 1}$ , so that  $a^2 \equiv -y^7 - y + 1 \pmod{\frac{y^{12} + 1}{2}}$ . Thus we have

$$\begin{aligned} 1 &= \left( \frac{-y^7 - y + 1}{(y^{12} + 1)/2} \right) = \left( \frac{y^7 + y - 1}{(y^{12} + 1)/2} \right) = \left( \frac{2(y^{12} + 1)}{y^7 + y - 1} \right) \\ &= \left( \frac{y^{12} + 1}{y^7 + y - 1} \right) = \left( \frac{y^6 - y^5 - 1}{y^7 + y - 1} \right) = \left( \frac{y^7 + y - 1}{y^6 - y^5 - 1} \right) \\ &= \left( \frac{y^5 + 2y}{y^6 - y^5 - 1} \right) = \left( \frac{y^4 + 2}{y^6 - y^5 - 1} \right) = - \left( \frac{y^6 - y^5 - 1}{y^4 + 2} \right) \\ &= \left( \frac{2y^2 - 2y + 1}{y^4 + 2} \right) = \left( \frac{y^4 + 2}{2y^2 - 2y + 1} \right) = \left( \frac{7}{2y^2 - 2y + 1} \right) \\ &= \left( \frac{2y^2 - 2y + 1}{7} \right) \end{aligned}$$



which is possible only when  $y$  is congruent to 1, 4, or 0 modulo 7. This is impossible since  $y$  is an odd power of 5. This completes the proof of the lemma.

#### §4 Further application of Lemma 3.2

In this section we show how Lemma 3.2 can be used in a different direction, treating an old problem which has already received much attention (see introduction).

**Theorem 4.1** Let  $C$  be an even positive integer, and let  $PQ$  be the largest squarefree divisor of  $C$ , where  $P$  is chosen so that  $(C/P)^{1/2}$  is an integer. If the equation

$$x^2 + C = y^n \quad (4.1)$$

has a solution  $(x, y, n)$  with  $x$  and  $y$  nonzero integers divisible by at most one prime,  $(x, y) = 1$ ,  $n$  a positive integer, and  $(x, y, n) \neq (7, 3, 4)$  or  $(401, 11, 5)$ , then we must have either  $n = 3$  or

$$n|N = 2 \cdot 3^u h(-P) \langle q_1 - \left(\frac{-P}{q_1}\right), \dots, q_n - \left(\frac{-P}{q_n}\right) \rangle$$

Here  $u = 1$  or  $0$  according as  $3 < P \equiv 3 \pmod{8}$  or not,  $h(-P)$  is the lowest  $h$  such that  $\mathfrak{a}^h$  is principal for every ideal  $\mathfrak{a}$  in  $\mathbf{Q}(\sqrt{-P})$ ,  $\langle a_1, a_2, \dots, a_n \rangle$  is the least common multiple of the members of the set  $S = \{a_1, a_2, \dots, a_n\}$  when  $S \neq \emptyset$ ,  $\langle a_1, a_2, \dots, a_n \rangle = 1$  when  $S = \emptyset$ ,  $q_1 q_2 \dots q_n = Q$  is the prime factorization of  $Q$ , and  $\left(\frac{a}{q}\right)$  is the familiar Legendre symbol unless  $q = 2$  in which case  $\left(\frac{a}{2}\right) = 0$ .

Proof: It suffices to prove the theorem for the case in which  $y$  is a positive prime. Assume there exists a solution to (4.1). Let  $\mathfrak{p}\bar{\mathfrak{p}}$  be the prime ideal factorization of  $y$  in  $\mathbf{Q}(\sqrt{-P})$ . Let  $k$  be the smallest number such that  $\mathfrak{p}^k = [\alpha]$  is principal with a generator  $\alpha$  having integer coefficients. When  $P = 1$ , we choose  $\alpha$  so that the coefficient of its imaginary term is even. When  $P = 3$  we can take  $k = 1$ . Then

$$\alpha^{n/k} = \pm x \pm \sqrt{-C}$$

where the  $\pm$  signs are independent. Note that when  $P = 3$  and  $\alpha^{n/k} \epsilon = x \pm \sqrt{-C}$  for some unit  $\epsilon$ , we must have  $\epsilon = \pm 1$ . Let  $j$  be the least number such that  $\alpha^j = u + vQ\sqrt{-P}$  for some integers  $u$  and  $v$ . By elementary properties of the coefficients of powers of integers in a quadratic field,  $jk|N/2$ . Also,  $jk|n = jkr$  for some  $r$ . So we have

$$(u + vQ\sqrt{-P})^r = \pm x \pm \sqrt{-C}$$

If  $r = 1$  or  $r = 2$ , the Theorem holds, so assume  $r \geq 3$ .

If  $r$  is even, then any prime dividing  $u$  must divide  $C$ , since  $\pm x \pm \sqrt{-C}$  must be divisible by  $(u+vQ\sqrt{-P})^2$ . Since  $(u, C) = 1$ , we must have  $u = \pm 1$  when  $r$  is even.

If  $r$  is odd, then  $u$  divides  $x$ .  $x = \pm 1$  implies  $u = \pm 1$ . Assume  $|x| > 1$ . Let  $x = \pm g^s$  where  $g$  is a positive prime and  $s > 0$ . Then, when  $r$  is odd,  $u = \pm g^t$  for some  $t \geq 0$ . Also, every prime dividing  $v$  divides  $C$ . Thus, if  $t > 0$ , then by Theorem 1 of [Sc],  $r = 1$  which we already excluded.

So  $u = \pm 1$  regardless of the value of  $x$  or the parity of  $r$ . Letting  $D = v^2 Q^2 P$ , we have

$$(1 + \sqrt{-D})^r = \pm x \pm w\sqrt{-D}$$

for some positive integer  $w$ . If  $w = 1$ , we see from Lemma 3.2 that  $r = 3$  and  $j = k = 1$ , so that  $n = 3$  and the theorem holds.

So  $w > 1$ , and  $w$  is divisible only by primes dividing  $C$ . In what follows, we apply Lemmas 1–3 of [Sc]. We must have at least one prime  $r_1$  dividing  $C$  which also divides  $r$ . Thus we have

$$(1 + \sqrt{-D})^{r_1} = \pm x_1 \pm w_1 \sqrt{-D} \tag{4.2}$$

where  $w_1 | w$ . If  $r_1$  is odd, we have

$$\pm w_1 = r_1 - \binom{r_1}{3} D + \binom{r_1}{5} D^2 - \dots \pm D^{\frac{r_1-1}{2}}. \tag{4.3}$$

$r_1 | w_1$ , and, if  $r_1 > 3$ , then  $r_1^2 \nmid w_1$ . Also, when  $r_1 > 3$ ,  $(w_1/r_1, C) = 1$ , so that  $w_1 = \pm r_1$ .

If  $r_1 = 3$ , we must have  $w_1 = \pm 3^z$  for some  $z > 0$  so that  $D = 3^z + 3$ . Now  $1 + D$  is the norm of  $\alpha^j$  which equals  $y^{jk}$ . But  $1 + D = 3^z + 4$  cannot be a perfect power of  $y$  by Lemma 2 of [ScSt]. So  $j = k = 1$ . Now  $|x_1| = 3D - 1 > 1$ . Also,  $(x_1, C) = 1$  so  $2 \nmid r$ . Thus,  $x_1$  must be a power of the prime dividing  $x$ . By Theorem 1 of [Sc],  $r = r_1$ ,  $n = 3jk = 3$ , and the theorem holds.

If  $r_1 = 5$  then (4.3) shows that  $\pm 5 = 5 - 10D + D^2$ . Since  $5 | D$ , this implies  $D = 10$ ,  $y^{jk} = 11$  which gives  $(x_1, y, r_1, j, k) = (401, 11, 5, 1, 1)$ . If  $r > r_1$ , we must have  $2 \nmid r$  and  $401 | x$ , so Theorem 1 of [Sc] shows  $r = r_1$ . This leads to the case  $(x, y, n) = (401, 11, 5)$ .

If  $r_1 \geq 7$ , (4.3) is impossible for  $w_1 = \pm r_1$ .

Finally, if  $r_1 = 2$  is the only prime dividing both  $r$  and  $C$ , then we have (4.2) with  $r_1 = 2$ ,  $|x_1| = D - 1$ . If  $D > 2$ , then, since  $D - 1 > 1$ ,

we have  $2 \nmid r/r_1$ . Therefore,  $D - 1 \mid x$  and we can use Theorem 1 of [Sc] as before to obtain  $r_1 = r$ . But this contradicts  $r > 2$ . So  $D = 2$ ,  $C = 2^t$  for some positive integer  $t$ . Since  $2 = r_1 \mid n$ , (4.1) implies  $y^{n/2} = 2^{t-2} + 1$ . If  $n = 2$  the theorem holds, so assume  $n > 2$ . It is a familiar elementary result that we must have  $y = 3$ ,  $n = 4$ , and  $t = 5$  which leads to the exceptional case  $(x, y, n) = (7, 3, 4)$ .

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